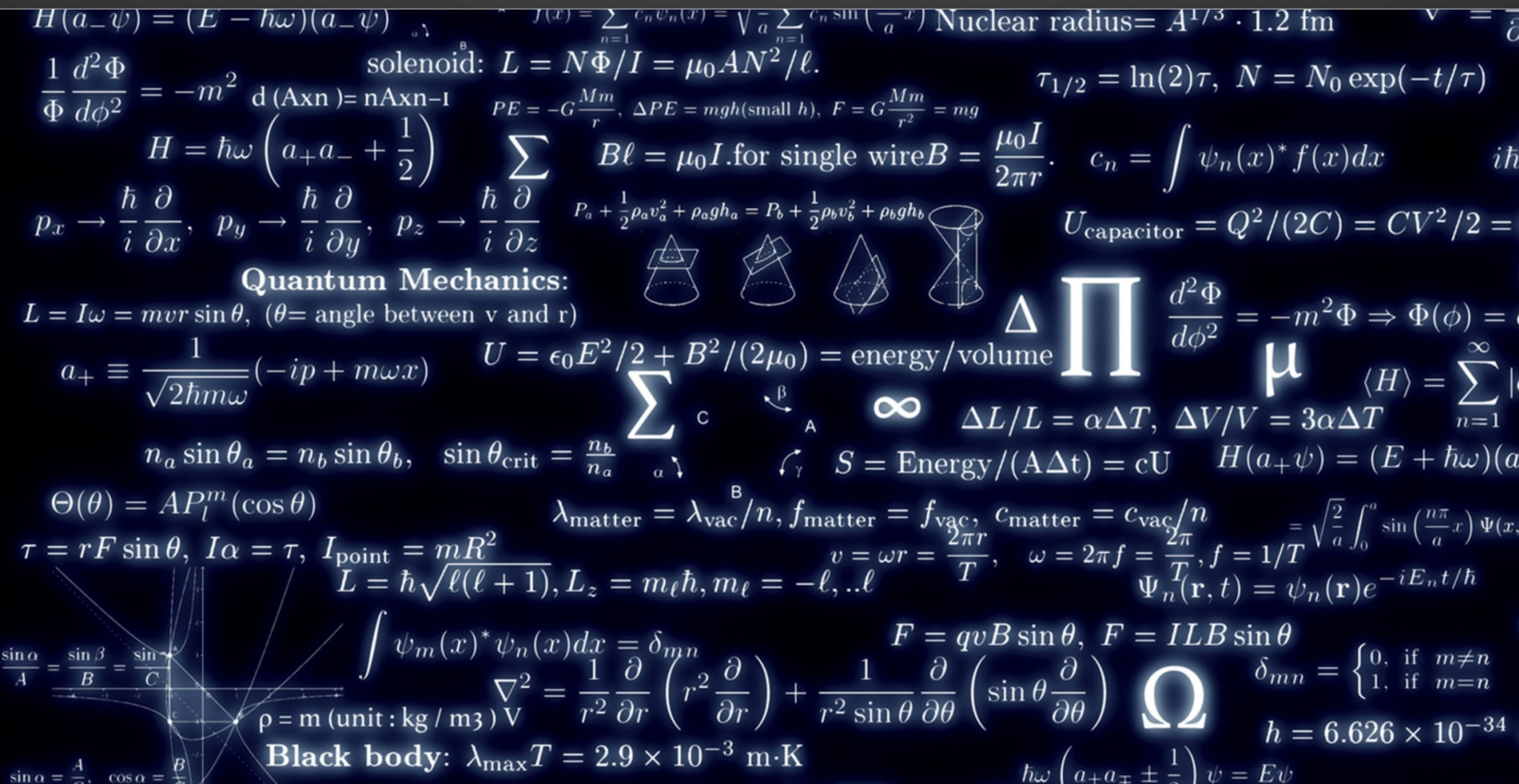
Article III

K^C PRIME NUMBERS



On the number of composite numbers less than a given value. Lemmas, continued.

Paper III: *What do we know about prime numbers distribution?*

Paper II presented 3 of 7 lemmas that confirm the conjecture introduced in paper I:

Suppose k is the amount of prime numbers between n and 2n+ 1, while c is the amount of prime numbers between n^2 and $(n+1)^2$

Theorem:
Between n^2 and $(n+1)^2$ exist c prime numbers,
c converges asymptotically to the value ≈ k

In this paper we present another two lemmas that are preceded with an essay part. Thus, the title contains the following question:

What do we actually know about prime numbers distribution?

The list is quite short: for most people, the issue of prime numbers distribution is limited to several proven facts and a vast collection of hypotheses.

1. Euclid: ***the theorem of infinite amount of prime numbers***

2. The Sieve of Eratosthenes: ***the algorithm for finding prime numbers***

3. C.F. Gauss: ***The Prime Number Theorem (PNT)***

4. P. Chebyshev: the proof of Bertrand's postulate – ***"There is at least one prime number between n and 2n",*** **improved by: P. Erdös –** ***"There are at least two prime numbers between n and 2n"***

Mathematicians are still impressed with the elegance of the reasoning attributed to Euclid. Firstly: Euclid has formulated a proper question; secondly: he has created a proper tool to build the proof. Let's start with a question formulated by Euclid: if all composite numbers are products of prime numbers, then is there a finite amount of prime numbers? In other words: is there a limited list of prime numbers that allows to obtain all composite numbers by multiplying them mutually?

The Euclide's tool let us discover that ***products of subsequent prime numbers increased by 1 divided by those prime numbers, or their products, give results with a remainder of 1.*** **Let's try to follow his reasoning, starting with $2\times3\times5+1=31$. So if the result of 31 is divided by 2, by 3, or by 5, or any mutual product of these numbers, the result will always have a remainder of 1. Why? Every product $\geq$ 30**

must contain a multiplication of $2\times5=10$, giving a result ending with 0. Thus, every product of subsequent prime numbers obtained this way, increased by one, when divided by any prime number different than 2 or 5, should give a result ending with digit 0, and also the remainder of 1. Conclusion? Either 31 is a prime number, or 31 is a composite number divisible by a prime number greater than 5.

Of course, 31 is a prime number, and two subsequent multiplications give the results that are also prime numbers: $2\times3\times5\times7+1=211$, $2\times3\times7\ 5\times7\times11+1=2311$, but the next multiplication: $2\times3\times5\times7\times11\times13+1=30031$ gives a composite number that is a product of two primes – $30031=59\times509$ – greater than 2, 3, 5, 7, 11 and 13. Since such procedure can be repeated infinitely, the conclusion is obvious: the amount of prime numbers must be infinite. It must be emphasised that Euclid worked using only his mind, and maybe... an abacus (in Greek maths of that time, the greatest named number was myriad, that is 10,000, then were only "plenty" of myriads).

Nevertheless, Eratosthenes, introducing his algorithm, allowed to move the limit of cognition of the distribution of prime numbers far beyond that. He proposed a "sieve" that could separate the chaos of prime numbers from the chaos of their consecutive multiplicities. If used in a bit different way than it was intended by its creator, it allows to postulate a new research perspective.

The impressive efforts of Euler and Gauss are worth noting. A significant part, or maybe even most of their works concerned the issue of finding the patterns in the prime numbers distribution. Euler's results were extraordinary, which for some researchers was the proof of close relationships of the shape of the surrounding physical world and the distribution of primes. It turned out that multiplication of certain prime numbers gives a result that is surprisingly related with the physical world - the wheel shape:

$$\frac{2^2}{2^2-1}\times\frac{3^2}{3^2-1}\times\frac{5^2}{5^2-1}\times\frac{7^2}{7^2-1}\times\frac{11^2}{11^2-1}\times\frac{13^2}{13^2-1}\times\frac{17^2}{17^2-1}\times\frac{19^2}{19^2-1}\times\frac{23^2}{23^2-1}\times\frac{29^2}{29^2-1}\times\frac{31^2}{31^2-1}\times\frac{37^2}{37^2-1}\times\dots=\frac{\pi^2}{6}.$$

A slight modification of this equation, that is replacing exponent 2 with variable x, gives the well-known Riemann ζ(x) function:

$$\frac{2^x}{2^x-1}\times\frac{3^x}{3^x-1}\times\frac{5^x}{5^x-1}\times\frac{7^x}{7^x-1}\times\frac{11^x}{11^x-1}\times\frac{13^x}{13^x-1}\times\frac{17^x}{17^x-1}\times\frac{19^x}{19^x-1}\times\frac{23^x}{23^x-1}\times\frac{29^x}{29^x-1}\times\frac{31^x}{31^x-1}\times\frac{37^x}{37^x-1}\times\dots.$$

But it was Gauss's study that provided us with the essential knowledge concerning the distribution of prime numbers. Throughout the years of his research work, he analysed prime numbers charts in search for any regularity that could illustrate the distribution of these numbers. In the course of his work, he formulated a vital question concerning the statistical analysis of their amount – it was the question of *the probability of occurrence of a prime number within a particular interval.*

Among the first hundred of natural numbers are 25 primes, so the probability of occurrence of a prime number is $\frac{1}{4}$. In the first thousand, the amount of prime numbers is 168,

thus the probability of occurrence of a prime number is $\cong \frac{1}{6}$. In 10,000, there are 1239 prime numbers, giving the probability of $\cong \frac{1}{8}$.

Respectively, for 100,000 the probability of occurrence of prime numbers is $\cong \frac{1}{10}$, while for 1,000,000 it is $\cong \frac{1}{12}$, etc. This particular statistical regularity, slightly rounded in the presentation above, has been denoted with a more precise mathematical formula by Gauss (without a proof, though). It is referred to as the *Prime Number Theorem (PNT):*

$$\pi(N) = \frac{N}{\log N}.$$

The fact that the statistical regularity noticed by Gauss could be proved only after a century clearly indicates the scale of difficulties that mathematicians have to cope with in proving the theorems concerning the distribution of prime numbers, but it also *shows the importance of a well-formulated research hypothesis*: throughout the centuries, the most outstanding minds of the times have been encouraging the development of maths working on the proof of this hypothesis.

However, the presentations concerning the issues of prime numbers contain certain stereotypes and reasoning patterns that make the heuristic explanation difficult. The introductions of the said presentations draw attention with the emphasised opinions like: "it is easy to notice that as natural numbers N increase, the amount of prime numbers decreases very quickly". Since we have acknowledged that Euclid has proved that *there is an infinite amount of prime numbers, then as N increases, the amount of prime numbers must also increase, up to infinity.* The quantitative statistics of that process, determined by Gauss, indicates only the differentiation of its dynamics: natural numbers generate much faster than their subset, that is the prime numbers. It leads to an obvious question: *is it possible to determine such subsets of natural numbers, in which the rate of increase is lower than the rate of increase of prime numbers, but in a way that would allow to perform an effective research process?* Well, *the squares of natural numbers meet this criterion perfectly.* It has already been noticed by Euler: *"primes are not as rare as the squares".*

In 1978, in a scope similar to Gauss – also without a proof – *another statistical regularity was indicated*: D. Andrica formulated a conjecture leading to the following conclusion, identical to the Legendre's conjecture: *"There is a prime number between the squares of two subsequent natural numbers".* The mathematicians have agreed that this conjecture, though it seems right, is weaker than the Oppermann's conjecture, formulated earlier. Another stereotype has been fixed: a weak conjecture.

Such valuation of that conjecture is stigmatising: an outstanding mathematician (physicist, geneticist, etc.) has proved a weak conjecture? Even amateurs in maths set themselves more ambitious goals. Nevertheless, it turns out that *the Legendre's conjecture is a key one, and after a deeper analysis it allows to formulate a model presenting a mechanism of primes distribution*. Why? Consider the fact that... *all prime numbers are situated between the squares of natural numbers.* Note that the square of any number, that is n^2, must be a composite number, since it has more than two divisors: it divides by 1,

by itself - that is n^2 and also by n. It leads to an obvious conclusion: *all primes must be distributed between the squares of subsequent natural numbers!* Thus, the only issue that should be explained is: *what is the frequency of occurrence of prime numbers between consecutive squares of natural numbers...* Surprisingly, the results of Gauss and Montgomery convinced mathematicians to the claim that primes are distributed quite regularly, in general. However, the statistical rule observed by Legendre would indicate that this regularity also occurs in much shorter intervals than it is indicated in the PNT. Consider the following statistical dependence: the interval from 0 to 100 contains 10 squares of numbers (1^2 to 10^2), for which there are 25 primes, which gives on average 2.5 prime numbers per each square. Respectively: in 10,000 there are 1229 prime numbers and 100 squares, which gives on average 12.29 prime numbers per each square.

The values increase: 10^8 contains respectively 10,000 squares and 5,761,455 primes, which means that for every square there are on average 576.1 prime numbers, while for 10^{18} there are 10^9 squares and 24,739,954,287,740,860 prime numbers, which gives 24,739,954.3 primes per square.

How could 25,000,000 millions of prime numbers in a certain interval between two squares migrate to another intervals? The researchers refer to a certain argument, but as we will show in further publications – Papers IV and V – it is only important locally and does not determine the amount of primes between n^2 and $(n+1)^2$.

In conclusion, we have a very strong hypothesis concerning the distribution of prime numbers in short intervals (squares of numbers), that reaches far beyond the Legendre's and Andrica's assumptions: *between the squares of consecutive primes, there is not only a certain prime number, but the amount of primes increases following a pattern that is a functional dependence*.

In conjunction with the conclusions resulting from the analysis of the algorithm of Eratosthenes, we may formulate a hypothesis which has a cascade functioning mechanism that will be presented in the following papers.

The Bertrand's postulate formulated in 1845, assuming the occurrence of at least one prime number between n and 2n, also indicates the existence of at least one prime number between 100 and 200. Although Erdös has demonstrated that there must be two prime numbers, the actual results obtained within the last 170 years reflect most evidently the frustration of the researchers: between 100 and 200 there are 21 primes, while between 100,000 and 200,000 there are 8392 of them, and the amount increases further to infinity.

Therefore, what could be previously proved about the distribution of prime numbers, departs dramatically from the actual state.

Certainly, the statistical correlation may be only a strong rationale to develop hypotheses - in science, and in particularly in mathematics, the most important goal is to find a single unique

cause and effect relationship. In order to find it, we must complete all necessary instruments, including the tools used in elementary and even intuitive maths.

The tools in mathematics are theorems. If they are used in evidentiary reasoning, they are called lemmas. Three of them have already been introduced in Paper II. Another two are presented below.
Suppose $n \in N$

<u>Lemma 4:</u> **To determine the first divisors of each composite number $< (n+1)^2$ it is enough to know prime numbers $\leq n$**

In fact, we have a new theorem, that as a lemma says that to determine all prime numbers between $10^2=100$ (n^2) and 11^2 ($(n+1)^2$) it is enough to know prime numbers ≤ 10 ($\leq n$). Thus, we can notice that each composite number <121 (that is $< (n+1)^2$) must be a subsequent multiplicity: 2, 3, 5 or 7 (multiplicity of prime number $\leq n$). Now we can use another tool available in school maths, formulating the following question:

How many numbers are exactly between $10^2=100$ (n^2) and 11^2 ($(n+1)^2$)?

The answer is:

2×10 ($2n$):

Suppose $n \in N$

<u>Lemma 2:</u> **The amount of numbers in an interval between n^2 and $(n+1)^2$ is $2n$.**

It is easy to notice that:

$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

However, the sense of our question is the question about the amount of numbers between n^2 and $(n+1)^2$, so it is enough to subtract 1 to get the answer: $2n$.

At this point, considering the Bertrand's postulate mentioned above, we should particularly point out ***the recurring dependence: n and 2n – it was the key factor for finding the functional dependence in the distribution of prime numbers***