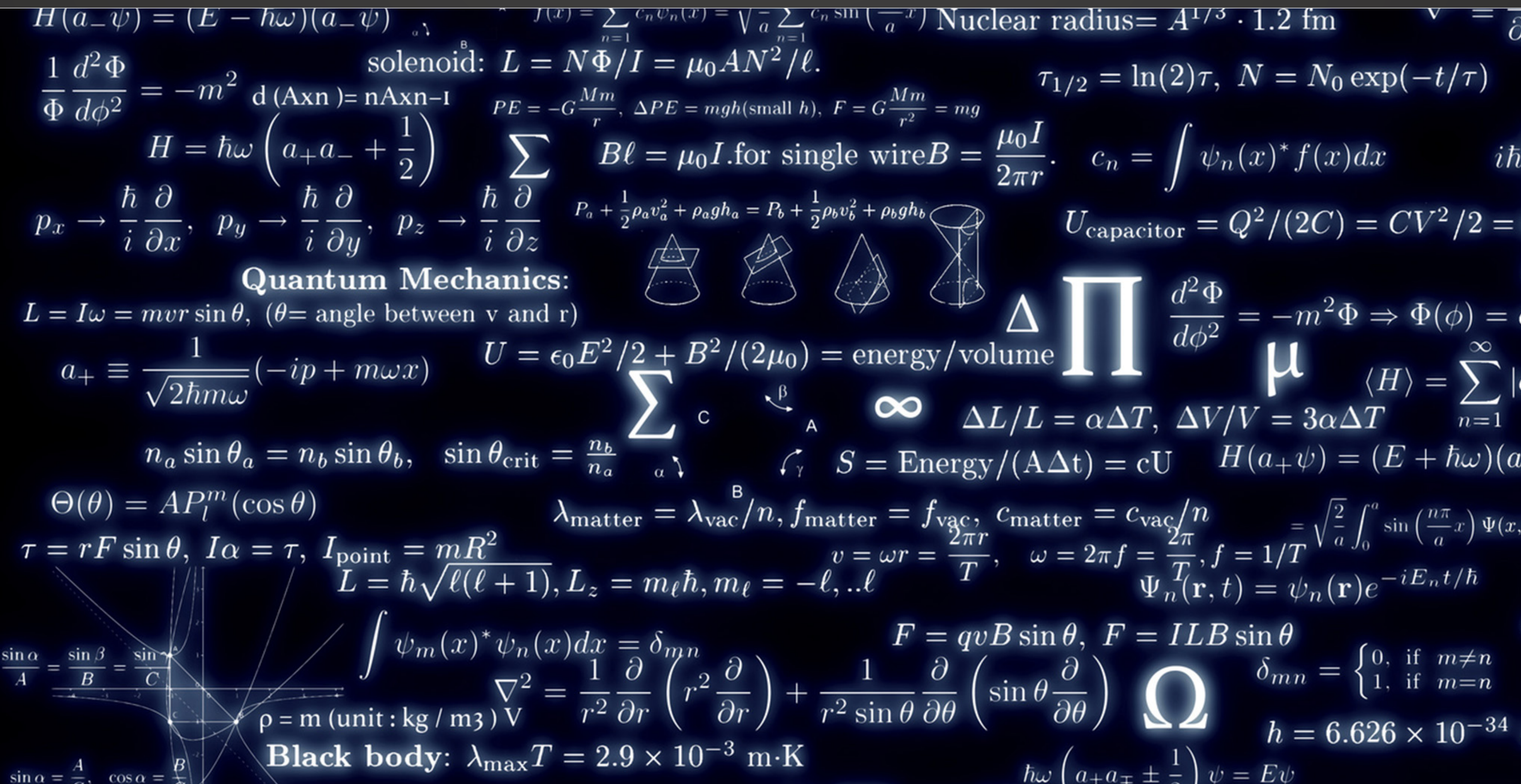
Article II

PRIME NUMBERS



On the number of composite numbers less than a given value. Lemmas.

Paper II: *A new research perspective in the prime number theory*

Let us start with recalling the theorem presented in Part I:

Below we propose a model presenting the cascade mechanism of prime numbers generation, formal notation of which is illustrated by the following theorem:
Suppose k is the amount of prime numbers between n and 2n+ 1, while c is the amount of prime numbers between n^2 and $(n+1)^2$

Theorem:

Between n^2 and $(n+1)^2$ exist c prime numbers,
c converges asymptotically to the value ≈ k

Of course, we come to the following question:

How important is this hypothesis?

Lemma 1:

Each prime number is situated between the squares of two natural numbers

The Legendre's hypothesis and the conclusion from Andrica's conjecture:

Theorem:

There is a prime number between the squares of two subsequent natural numbers

Adrien-Marie Legendre died 200 years ago, and the year 2018 will mark the 50th anniversary of the formulation of the Dorin Andrica's conjecture. Mathematicians around the world believe that this issue should be settled urgently. Thus, below is presented an attempt to develop a heuristic explanation, followed by a presentation of lemmas leading to a proof.

According to the Big Bang theory, this event happened about 14 billion years ago. The question of what existed before has so far been left to the unorthodox inquiries of cosmologists. Researchers have been searching for answers concerning its structure, and as a consequence, the way it works. So when in 1972, a certain mathematician and a physicist decided to talk about their work over a cup of tea at the Princeton University canteen, they didn't expect that their conversation would galvanise the entire micro-cosmos of the scientific community.

Montgomery and Dyson shocked not only mathematicians and physicists, but also the philosophy of science: it turned out that the distances of non-trivial zeros of the Riemann Zeta Function, ζ(x), and as a consequence, perhaps, the order of prime numbers, as well as the consecutive nuclear energy levels of heavy elements, are described by the same elementary function:

$\left(\frac{\sin\pi u}{\pi u}\right)^2 \leftarrow$ *Riemann ζ(x) function's zero distribution function* and $\left[\frac{\sin\pi r}{\pi r}\right]^2 \leftarrow$ *heavy elements' nuclear energy level distribution function*

Mathematicians could finally *stop hiding their research on this hypothesis*, while the civilisation was entering the digital era. Digits denote numbers, among which prime numbers are of key importance. Prime numbers can be divided only by 1 and themselves. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 are the chaos of the first 25 prime numbers. However, if *we complement these prime numbers with their mutual products and two special numbers:*

0 and 1, we get the unique order of the world of all natural numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... and so on to infinity. To trivialise: *fine (natural) order: 101, 102, 103, 104 ... must be preceded by the chaos of primality 2, 3, 5, 7...*

In Greek, άtomos means: indivisible. Maybe this is the reason why researchers perceive prime numbers as atoms that build the world of numbers; while physicists, extrapolating the relations between prime numbers and natural numbers, postulate analogous relations in the chaos of the world of elementary particles, described by quantum mechanics. In these projections, also the order of the macro-world, described according to Newtonian mechanics, would have particular relations with the world of prime numbers. Therefore, explaining the mystery of the distribution of prime numbers is not only the most important problem of mathematics, but also a Great Undertaking for other scientific disciplines: the formulation of a general Theory of Everything. *We owe the functioning of our entire civilisation in its current shape to prime numbers:* the Internet, credit cards, money transfers, military and intelligence codes, are some of the most obvious examples. Those containing a hundred and more digits can earn a lot of money. They are stored in treasuries and guarded just like reserves of gold – many would love to buy them, and prime numbers *just determine the functioning of contemporary economic relations.* The

existence of an infinite amount of prime numbers was postulated around 300 B.C. by Euclid, who gathered the entire mathematical knowledge available at that time in his "Elements" and *offered mathematics to mathematicians.*

The tool that is used to find them also originated in the ancient times: it is believed that Eratosthenes defined the prime-number sieve. Theoretically, it enables one to find any, even very large, prime number, provided that you have... an infinite amount of time.

Two thousand years later, new results were obtained: Pierre de Fermat, called the prince of amateurs, proved the hypothesis known as Fermat's little theorem. An avalanche of great and small discoveries followed: Euler, Gauss, Chebyshev, Riemann and many other outstanding minds of their times, over the last three centuries, have strived to unveil the ancient mystery of prime numbers. Leonhard Euler was the first to suppose that nature owes its shape to prime numbers: "the Golden Key" to all mysteries of nature lies among these numbers. C.F. Gauss – the prince of mathematicians – used an equation to make a quite good estimation of the amount of prime numbers. "If mathematics is the queen of science (in fact it is), then the queen of mathematics is the theory of numbers", and its royal problem lies in the above-mentioned non-trivial zeros of the Riemann ζ(x) function. It is commonly believed that it is the most significant intellectual challenge in human history. For instance, Riemann's work enabled Albert Einstein to formulate his theory of relativity.

However, the partiality of these discoveries does not provide generalisations like the contributions of Euclid and Eratosthenes made two thousand years ago. This fact, naturally, leads to researchers' frustration: in fact, we know very little about the distribution of prime numbers. It is therefore understandable that *every educated person, even with interests quite distant from mathematics, should have a certain perception of prime numbers and the progress of research on their distribution at level of popular science.* Of course we may argue about the results of mathematicians' work. Every general description of that research would encompass several hundreds of the most important hypotheses. Some of them have been proved successfully, while others will be verified in the future. High rewards are set for proof of certain important theorems. However, the knowledge on *the distribution of prime numbers sill remains a mystery for the researchers, although a significant part of mathematics, in form of <u>numerous theorems</u>, has been formulated with the assumption of truthfulness of the Riemann's hypothesis.* That fact, of course, only deepens the frustration of researchers: *we do not even know what would have to be proven in order to resolve the problem.* This frustration is reflected in the researchers' question: *"Is there a way to understand [how prime numbers are distributed], if not completely, then at least to the degree that would allow one to determine the pattern that they impose on mathematics?"*

This was followed by a period of waiting for a new, revolutionary idea that could indicate the path for further research. There are different opinions: some researchers claim that a solution

of the problem would set the measure of maturity of humanity, which should be understood as the maturity of mathematics as practised by our civilisation. However, there are researchers who accept the possibility of a *random solution*, available even to amateurs.

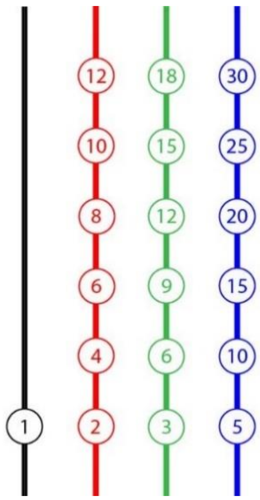
But if examining prime numbers themselves does not bring the expected results, maybe the research perspective should be widened? To illustrate this, let's analyse the following example: let's place two observers, A and B observing a set of segments, in two different points:

Observer A, with the assumption of a certain idealisation, will perceive and describe the model of this set as a unicolour point, while Observer B will construct a model of perception as a set of 11 segments with different colours and at different distances to each other. The analogy to segments is not incidental – a physical model of the distribution of prime numbers may be illustrated by a question in the field of measurement physics: why can segments with certain lengths be measured only by a unit segment(prime numbers), while the others are by segments longer than the unit segments (composite numbers)? There is no doubt that the problem of distribution of prime numbers is a challenge for science: maybe satisfactory progress could be reached by creating not a single model, but two - (or more) models based on several research perspectives? Below we suggest expanding the research perspective, by trying to give an answer to the following question:

What do composite numbers actually tell us about the distribution of prime numbers?

Before we discuss the mathematical part of the reasoning, we would like to analyse the model that we use every day. Let's look at an example, the map of London Underground: all stations are marked with points along 17 coloured lines. Unwinding these lines and arranging vertically beside each other will give the objects that mathematicians call topological braids. If we replace the names of the stations along the red line with the numbers 2, 4, 6, ..., along the green line - 3, 6, 9, ... and along the blue line - 5, 10, 15, ..., we can start constructing a pattern for natural numbers that is determined by prime numbers.

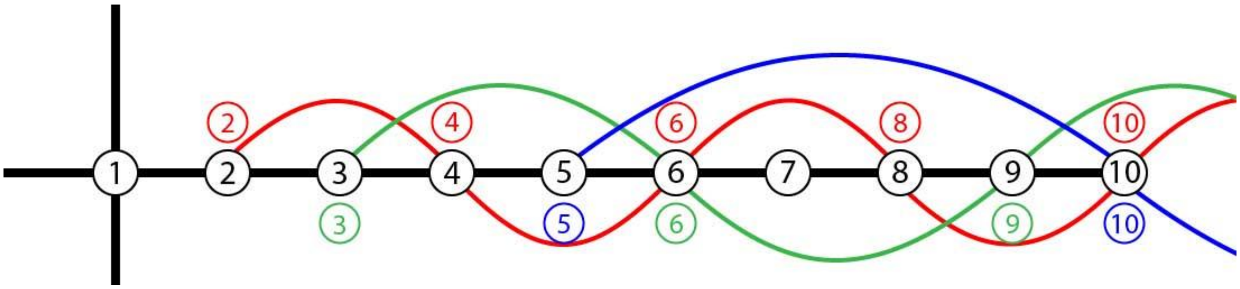
A perfect physical intuition for our topological braids may be four stretchy rubber bands with strung numbered beads: on the first rubber band (black), only one bead with number 1. On the second rubber band (red) - beads with doubles of 2, on the third (green) - beads with triples of 3, while on the fourth (blue) - beads with subsequent multiplications of number 5. Here is the model we have obtained:



On a straight line, in equal distances, let's mark the following points: 1, 2, 3, 4, 5, 6,... etc.:

0____1____2____3____4____5____6____7____8____9____10____11___

Now let's interweave the lines:



As it can easily be noticed, this topological analogy of prime numbers distribution among natural numbers and a map of any underground system, is striking. Now let's repeat our question: *What do composite numbers actually tell us about the distribution of prime numbers?* Well, it turns out that we can describe the set of natural numbers quite well analysing its complement of composite numbers. Why? We have already mentioned that the sums of these two sets along with two special numbers (0 and 1) give the set of all natural numbers. But what is even more significant: *all natural numbers can be arranged in arithmetic sequences in which prime numbers are the first terms of these sequences, respectively, while the difference* r *in these sequences, is equal to these prime numbers:*

Lemma 2:

	:		:		:		:		:		:
	:		:		:		:		:		:
2×6	12	3×6	18	5×6	30	7×6	42	11×6	66	13×6	78
2×5	10	3×5	15	5×5	25	7×5	35	11×5	55	13×5	65
2×4	8	3×4	12	5×4	20	7×4	28	11×4	44	13×4	52
2×3	6	3×3	9	5×3	15	7×3	21	11×3	33	13×3	39
2×2	4	3×2	6	5×2	10	7×2	14	11×2	22	13×2	26
	↑ r = 2		↑ r = 3		↑ r = 5		↑ r = 7		↑ r = 11		↑ r = 13
0 1	2 $a_1 = 2$		3 $a_1 = 3$		5 $a_1 = 5$		7 $a_1 = 7$		11 $a_1 = 11$		13 $a_1 = 13$

↑

Perspective of Observer C

Chart 1

In contrast, *the ratio of the distances* (fractions below) e.g. the tenth, hundredth, thousandth *multiplication of prime number* 2 and, respectively, the tenth, hundredth, thousandth, etc. multiplication of prime number 3, *is the conservation of the ratio of distances of these prime numbers.*

$$\frac{2}{3}=\frac{20}{30}=\frac{200}{300}\dots\ \frac{3}{5}=\frac{30}{50}=\frac{300}{500}\dots\ \frac{5}{7}=\frac{50}{70}=\frac{500}{700}\dots\ \ \frac{7}{11}=\frac{70}{110}=\frac{700}{1100}\dots\ \ \frac{11}{13}=\frac{110}{130}=\frac{1100}{1300}=\cdots$$

The mutual proportions (fractions) of composite numbers presented above, are identical as the ones obtained for prime numbers. It is easy to notice that the obtained analogy is the intriguing analogy to fractal structures. Thus, the following conclusion seems obvious: *the distribution of composite numbers is only the multiplied chaos that is the functional representation of the chaos of prime numbers distribution.* Thus, above all, we have prime numbers as the indivisible atoms of the world of natural numbers. Their consecutive multiplications form topological braids that interweave in points of repeating numerical values, e.g. 6, 10, 12, 14, 15, etc. These interweaves join them in the set of natural numbers. This way we have obtained the perspective of Observer C. This perspective actually allows to change the approach to the distribution of prime numbers: composite numbers may be perceived only as consecutive multiplications of prime numbers (Observer C), or (perspective of Observer B) composite numbers may be perceived only as mutual products of prime numbers:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 ...

The perspective of Observer C is simply unwinding of these topological braids that are the postulated arithmetic sequences. As a result, we obtain a certain outline of the pattern that prime numbers force on natural numbers: *the set of prime numbers has a mutual feedback relation with its multiplications.*

Let's analyse the benefits of the proposed change of perspective.
The first one can be noticed immediately: *this change of research perspective does not distinguish number 2 in the set of prime numbers.*

Most theorems concerning prime numbers have been formulated for odd prime numbers, while number 2, being the only even prime number, has been treated almost mystically. Furthermore, the division of natural numbers into even numbers (multiplications of 2) and odd numbers enabled the Pythagoreans to discover irrational numbers, which undermined their founding myth: the world is natural numbers, and in reasonable cases, their quotient. In the changed perspective of the view on prime numbers, *each prime number has the same property: being the*

first term of a sequence with specific parameters, it is the only number that can divide <u>all</u> its multiplications - thus, the status of number 2 is equal to others. **Another certain benefit relates to the *proposal of building a model of prime numbers distribution that would be a functional dependence.***

Lemma 3:

The topological braids presented in the charts below, being numerical sequences in which the first term is prime number p_n, and the other terms are consecutive multiplications of p_n that do not contain prime numbers lesser than p_n, shall be called C-type sequences in the following part of the paper. Now we can return to the outline of the pattern that prime numbers force on natural numbers. Below, such sequence is illustrated using specific numerical examples. Topological braids are assigned with appropriate numerical values in a way presented in the two charts below (Chart 2/I and Chart 3/I below, respectively). It is easy to notice that in Chart 2/I, in order to be able to generate a braid of a consecutive prime number, that number cannot appear as a value in any of the preceding braids, e.g. between prime number 2 and its second multiplication $2 \times 2 = 4$ appears a slot for a new prime number, that is 3. Furthermore, between the second and the third multiplication of 2 (4 and 6) appears a slot for another prime number, 5. It is even more clearly illustrated in the following chart (with an already noticeable outline of functional distribution of prime numbers): (Chart 3/I) The data in this chart indicate that slots for new prime numbers appear between the squares of two consecutive natural numbers: between 1^2 and 2^2 appears number 3. Between 2^2 and 3^2 are two prime numbers: 5 and 7. Between 3^2 and 4^2 appear another prime numbers: 11 and 13, etc. Now we can see the outline of the pattern that prime numbers force on natural numbers. This pattern shall be further analysed in our following papers, but the in-depth analysis allows to formulate the following heuristic explanation: *the knowledge of the distribution of composite numbers, in a proper research perspective, provides essential knowledge on the distribution of prime numbers.* This procedure enabled proposing the *cascade model of the mechanism determining prime numbers distribution*. Furthermore: consequently, we are also able to present *another model describing the mechanism of generation of twin prime numbers and represent prime numbers distribution as a combination of several elementary functions, which in turn can be denoted using equally elementary formulas.* Using this model, we have managed to find all first degree polynomials of two variables that give only consecutive and all prime numbers. We have also been able to develop *the interpretation of the mechanism that determines the possibility of presentation of any even number as a sum of two prime numbers and supporting the Goldbach's conjecture*. Therefore, the presentation has been divided into relevant parts concerning the individual issues.

Tablica 2

2× 60 = 120				
2× 59 = 118				
2× 58 = 116				
2× 57 = 114				
2× 56 = 112				
2× 55 = 110				
2× 54 = 108				
2× 53 = 106				
2× 52 = 104				
2× 51 = 102				
2× 50 = 100				
2× 49 = 98				
2× 48 = 96				
2× 47 = 94				
2× 46 = 92				
2× 45 = 90				
2× 44 = 88				
2× 43 = 86				
2× 42 = 84				
2× 41 = 82				
2× 40 = 80				
2× 39 = 78	3× 39 = 117			
2× 38 = 76	-			
2× 37 = 74	3× 37 = 111			
2× 36 = 72	-			
2× 35 = 70	3× 35 = 105			
2× 34 = 68	-			
2× 33 = 66	3× 33 = 99			
2× 32 = 64	-			
2× 31 = 62	3× 31 = 93			
2× 30 = 60	-			
2× 29 = 58	3× 29 = 87			
2× 28 = 56	-			
2× 27 = 54	3× 27 = 81			
2× 26 = 52	-			
2× 25 = 50	3× 25 = 75			
2× 24 = 48	-			
2× 23 = 46	3× 23 = 69	5× 23 = 115		
2× 22 = 44	-	-		
2× 21 = 42	3× 21 = 63	-		
2× 20 = 40	-	-		
2× 19 = 38	3× 19 = 57	5× 19 = 95		
2× 18 = 36	-	-		
2× 17 = 34	3× 17 = 51	5× 17 = 85	7× 17 = 119	
2× 16 = 32	-	-	-	
2× 15 = 30	3× 15 = 45	-	-	
2× 14 = 28	-	-	-	
2× 13 = 26	3× 13 = 39	5× 13 = 65	7× 13 = 91	
2× 12 = 24	-	-	-	
2× 11 = 22	3× 11 = 33	5× 11 = 55	7× 11 = 77	
2× 10 = 20	-	-	-	
2× 9 = 18	3× 9 = 27	-	-	
2× 8 = 16	-	-	-	
2× 7 = 14	3× 7 = 21	5× 7 = 35	7× 7 = 49	
2× 6 = 12	-	-	-	
2× 5 = 10	3× 5 = 15	5× 5 = 25	-	
2× 4 = 8	-	-	-	
2× 3 = 6	3× 3 = 9	-	-	
2× 2 = 4	-	-	-	
↑	↑	↑	↑	↑
2	3	5	7	11

K^C PRIME NUMBERS

Tablica 3

2× 60 = 120				
2× 59 = 118			7× 17 = 119	
2× 58 = 116	3× 39 = 117		-	
2× 57 = 114	-	5× 23 = 115	-	
2× 56 = 112	-	-	-	
2× 55 = 110	3× 37 = 111	-	-	
2× 54 = 108	-	-	-	
2× 53 = 106	-	-	-	
2× 52 = 104	3× 35 = 105	-	-	
2× 51 = 102	-	-	-	
2× 50 = 100	-	-	-	
2× 49 = 98	3× 33 = 99	-	-	
2× 48 = 96	-	-	-	
2× 47 = 94	-	5× 19 = 95	-	
2× 46 = 92	3× 31 = 93	-	-	
2× 45 = 90	-	-	7× 13 = 91	
2× 44 = 88	-	-	-	
2× 43 = 86	3× 29 = 87	-	-	
2× 42 = 84	-	5× 17 = 85	-	
2× 41 = 82	-	-	-	
2× 40 = 80	3× 27 = 81	-	-	
2× 39 = 78	-	-	-	
2× 38 = 76	-	-	7× 11 = 77	
2× 37 = 74	3× 25 = 75	-	-	
2× 36 = 72	-	-	-	
2× 35 = 70	-	-	-	
2× 34 = 68	3× 23 = 69	-	-	
2× 33 = 66	-	-	-	
2× 32 = 64	-	5× 13 = 65	-	
2× 31 = 62	3× 21 = 63	-	-	
2× 30 = 60	-	-	-	
2× 29 = 58	-	-	-	
2× 28 = 56	3× 19 = 57	-	-	
2× 27 = 54	-	5× 11 = 55	-	
2× 26 = 52	-	-	-	
2× 25 = 50	3× 17 = 51	-	-	
2× 24 = 48	-	-	7× 7 = 49	
2× 23 = 46	-	-	-	
2× 22 = 44	3× 15 = 45	-	-	
2× 21 = 42	-	-	-	
2× 20 = 40	-	-	-	
2× 19 = 38	3× 13 = 39	-	-	
2× 18 = 36	-	-	-	
2× 17 = 34	-	5× 7 = 35	-	
2× 16 = 32	3× 11 = 33	-	-	-
2× 15 = 30	-	-	-	
2× 14 = 28	-	-	-	
2× 13 = 26	3× 9 = 27	-	-	
2× 12 = 24	-	5× 5 = 25	-	-
2× 11 = 22	-	-	-	
2× 10 = 20	3× 7 = 21	-	-	
2× 9 = 18	-	-	-	
2× 8 = 16	-	-	-	
2× 7 = 14	3× 5 = 15	-	-	
2× 6 = 12	-	-	-	
2× 5 = 10	-	-	-	
2× 4 = 8	3× 3 = 9	-	-	
2× 3 = 6	-	-	-	
2× 2 = 4	-	-	-	
↑	↑	↑	↑	↑
2	3	5	7	11

K^C PRIME NUMBERS